Your Name Here
Math 220
HW # 7
November 13, 2018

Exercise 1. *A sequence* $\{a_n\}$ *is defined recursively by* $a_1 = 1$, $a_2 = 2$, *and* $a_n = a_{n-1} + 2a_{n-2}$ *for* $n \geq 3$. *Conjecture a formula for* a_n *and verify that your conjecture is correct.*

Proof. The sequence continues with

$$a_3 = 4, \quad a_4 = 8, \quad a_5 = 16, \cdots$$

so it looks like the formula should be $a_n = 2^{n-1}$. We proceed to prove this with strong induction.

Base Case ($n = 1$)

$$2^{1-1} = 2^0 = 1 = a_1$$

so the base case is verified.

Ind. Hyp. Assume that for all $1 \leq i \leq k$, that $a_k = 2^{k-1}$.

Ind. Step If $k = 2$, the recursion relation does not yet apply, so we check that separately:

$$2^{2-1} = 2^1 = 2 = a_2$$

so the formula holds for $k = 2$. Thus we can assume that $k \geq 3$. So, by the recursion relation and inductive hypothesis, we have

$$a_{k+1} = a_k + 2a_{k-1} = 2^{k-1} + 2 \cdot 2^{k-1-1} = 2^{k-1} + 2^{k-1} = 2^k$$

as desired.

□

Exercise 2. *A sequence* $\{a_n\}$ *is defined recursively by* $a_1 = 1$, $a_2 = 4$, $a_3 = 9$, *and* $a_n = a_{n-1} - a_{n-2} + a_{n-3} + 2(2n - 3)$ *for* $n \geq 4$. *Conjecture a formula for* a_n *and verify that your conjecture is correct.*

Proof. The sequence continues with

$$a_4 = 16, \quad a_5 = 25, \quad a_6 = 36, \cdots$$

so it looks like the formula should be $a_n = n^2$. We proceed to prove this with strong induction.

Base Case ($n = 1$)

$$1^2 = 1 = a_1$$

so the base case is verified.

Ind. Hyp. Assume that for all $1 \leq i \leq k$, that $a_k = k^2$.

Ind. Step If $k = 2$ or 3, the recursion relation does not yet apply, so we check those separately:

($k = 2$)

$$2^2 = 4 = a_2$$

($k = 3$)

$$3^2 = 9 = a_3$$

so the formula holds for $k = 2, 3$. Thus we can assume that $k \geq 4$. So, by the recursion relation and inductive hypothesis, we have

$$\begin{aligned} a_{k+1} &= a_k - a_{k-1} + a_{k-2} + 2(2(k+1) - 3) \\ &= k^2 - (k-1)^2 + (k-2)^2 + 2(2k-1) \\ &= k^2 - k^2 + 2k - 1 + k^2 - 4k + 4 + 4k - 2 \\ &= k^2 + 2k + 1 = (k+1)^2 \end{aligned}$$

as desired.

□

Exercise 3. *Answer the following questions:*

(a) *Is* $3 \in \{1, 2, 3\}$*?*
(b) *Is* $1 \subset \{1\}$*?*
(c) *Is* $\{2\} \in \{1, 2\}$*?*
(d) *Is* $\{3\} \in \{1, \{2\}, \{3\}\}$*?*
(e) *Is* $1 \in \{1\}$*?*
(f) *Is* $\{2\} \subset \{1, \{2\}, \{3\}\}$*?*
(g) *Is* $\{1\} \subset \{1, 2\}$*?*
(h) *Is* $1 \in \{\{1\}, 2\}$*?*
(i) *Is* $\{1\} \subset \{1, \{2\}\}$*?*
(j) *Is* $\{1\} \subset \{1\}$*?*

Solution.

(a) Yes
(b) No
(c) No
(d) Yes
(e) Yes
(f) No
(g) Yes
(h) No
(i) Yes

(j) Yes

□

Exercise 4. *Write each of the following sets by listing its elements within braces.*

(a) $A = \{n \in \mathbb{Z} | -4 < n \leq 4\}$
(b) $B = \{n \in \mathbb{N} | n^3 < 100\}$
(c) $C = \{x \in \mathbb{R} | x^2 - x = 0\}$
(d) $D = \{4n | n \in \mathbb{Z}\}$

Solution.

(a) $A = \{n \in \mathbb{Z} | -4 < n \leq 4\} = \{-3, -2, -1, 0, 1, 2, 3, 4\}$
(b) $B = \{n \in \mathbb{N} | n^3 < 100\} = \{1, 2, 3, 4\}$
(c) $C = \{x \in \mathbb{R} | x^2 - x = 0\} = \{0, 1\}$
(d) $D = \{4n | n \in \mathbb{Z}\} = \{..., -12, -8, -4, 0, 4, 8, 12, ...\}$

□

Exercise 5. *Let* $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ *be the universal set and let* $A = \{3, 6, 9\}$, $B = \{3, 4, 5, 6, 7\}$, *and* $C = \{4, 5, 7\}$. *Find the following:*

- $A \cup B$
- $A \cap B$
- $A \cap C$
- B^c
- $B \setminus C$
- $U \cup A$
- $B \cap U$

Solution.

- $A \cup B = \{3, 4, 5, 6, 7, 9\}$
- $A \cap B = \{3, 6\}$
- $A \cap C = \varnothing$
- $B^c = \{1, 2, 8, 9\}$
- $B \setminus C = \{3, 6\}$
- $U \cup A = U$
- $B \cap U = B$

□

Exercise 6. *Let* $A = \{a, b, c\}$, $B = \{b, c, d\}$, *and* $C = \{b, c, e\}$.

(a) *Find* $A \cup (B \cap C)$, $(A \cup B) \cap C$, *and* $(A \cup B) \cap (A \cup C)$. *Which of these sets are equal?*
(b) *Find* $A \cap (B \cup C)$, $(A \cap B) \cup C$, *and* $(A \cap B) \cup (A \cap C)$. *Which of these sets are equal?*
(c) *Find* $A \setminus (B \setminus C)$ *and* $(A \setminus B) \setminus C$. *Are these sets are equal?*

Solution.

(a) $A \cup (B \cap C) = \{a, b, c\}$, $(A \cup B) \cap C = \{b, c\}$, and $(A \cup B) \cap (A \cup C) = \{a, b, c\}$. So we have $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(b) $A \cap (B \cup C) = \{b, c\}$, $(A \cap B) \cup C = \{b, c, e\}$, and $(A \cap B) \cup (A \cap C) = \{b, c\}$. So we have $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(c) $A \setminus (B \setminus C) = \{a, b, c\}$ and $(A \setminus B) \setminus C = \{a\}$. This shows these sets are not equal.

□

Exercise 7. *Answer the following questions and give explanations for your answers.*

(a) *Is* $\{\{a, d, e\}, \{b, c\}, \{d, f\}\}$ *a partition of* $\{a, b, c, d, e, f\}$*?*

(b) *Is* $\{\{w, x, v\}, \{u, y, q\}, \{p, z\}\}$ *a partition of* $\{p, q, u, v, w, x, y, z\}$*?*

(c) *Is* $\{\{3, 7, 8\}, \{2, 9\}, \{1, 4, 5\}\}$ *a partition of* $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$*?*

Solution.

(a) No, because the first and third sets overlap.

(b) Yes, every element is in some set in the collection, and the sets are pairwise disjoint.

(c) No, because no set in the collection contains 6.

□

Exercise 8. *Let* $\mathbb{Z}$ *be the set of all integers and let*

$$\begin{aligned} A_0 &= \{n \in \mathbb{Z} | n = 4k, \textit{for some integer } k\}, \\ A_1 &= \{n \in \mathbb{Z} | n = 4k + 1, \textit{for some integer } k\}, \\ A_2 &= \{n \in \mathbb{Z} | n = 4k + 2, \textit{for some integer } k\}, \textit{ and} \\ A_3 &= \{n \in \mathbb{Z} | n = 4k + 3, \textit{for some integer } k\}. \end{aligned}$$

Is $\{A_0, A_1, A_2, A_3\}$ *a partition of* $\mathbb{Z}$*? Explain your answer.*

Solution. Yes, it is a partition of $\mathbb{Z}$. When divided by 4, every integer will have a remainder of $0, 1, 2$, or 3. Each of the sets correspond to this remainder. Since no integer has two different remainders when divided by 4, these sets are disjoint as well. Therefore it is a partition. □

Exercise 9. *Let* $A = \{1, 2, 3, 4\}$ *and* $B = \{a, c\}$*. Compute the following:*

(a) $A \times B$

(b) $B \times A$

(c) $A \times A$

(d) $B \times B$

(e) $\varnothing \times B$

(f) $(A \times B) \times B$

(g) $A \times (B \times B)$

Solution.

(a) $A \times B = \{(1,a),(2,a),(3,a),(4,a),(1,c),(2,c),(3,c),(4,c)\}$

(b) $B \times A = \{(a,1),(a,2),(a,3),(a,4),(c,1),(c,2),(c,3),(c,4)\}$

(c)
$$\begin{aligned} A \times A &= \{(1,1),(2,1),(3,1),(4,1),(1,2),(2,2),(3,2),(4,2),\\ &\quad (1,3),(2,3),(3,3),(4,3),(1,4),(2,4),(3,4),(4,4)\} \end{aligned}$$

(d) $B \times B = \{(a,a),(a,c),(c,a),(c,c)\}$

(e) $\varnothing \times B = \varnothing$

(f)
$$\begin{aligned} (A \times B) \times B &= \{((1,a),a),((2,a),a),((3,a),a),((4,a),a),((1,c),a),((2,c),a),((3,c),a),((4,c),a),\\ &\quad ((1,a),c),((2,a),c),((3,a),c),((4,a),c),((1,c),c),((2,c),c),((3,c),c),((4,c),c)\} \end{aligned}$$

(g)
$$\begin{aligned} A \times (B \times B) &= \{(1,(a,a)),(2,(a,a)),(3,(a,a)),(4,(a,a)),(1,(c,a)),(2,(c,a)),(3,(c,a)),(4,(c,a)),\\ &\quad (1,(a,c)),(2,(a,c)),(3,(a,c)),(4,(a,c)),(1,(c,c)),(2,(c,c)),(3,(c,c)),(4,(c,c))\} \end{aligned}$$

$\square$

Exercise 10. *Compute the following power sets. We will denote the power set of A by $\mathscr{P}(A)$.*

(a) $\mathscr{P}(\{1,2,3,4\})$

(b) $\mathscr{P}(\varnothing)$

(c) $\mathscr{P}(\{\varnothing\})$

Solution.

(a)
$$\begin{aligned} \mathscr{P}(\{1,2,3,4\}) &= \{\varnothing,\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\},\\ &\quad \{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,3,4\}\} \end{aligned}$$

(b) $\mathscr{P}(\varnothing) = \{\varnothing\}$

(c) $\mathscr{P}(\{\varnothing\}) = \{\varnothing,\{\varnothing\}\}$

$\square$